

Weak dispersion for the Dirac equation on curved space-time

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Abstract. In this paper we prove local smoothing estimates for the Dirac equation on some non-flat manifolds; in particular, we will consider asymptotically flat and wrapped products metrics. The strategy of the proofs relies on the multiplier method.

1. INTRODUCTION

The *Dirac equation on \mathbb{R}^{1+3}* is a constant coefficient, hyperbolic system of the form

$$(1.1) \quad iu_t + \mathcal{D}u + m\beta u = 0$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{C}^4$, $m \geq 0$ is called the *mass*, the *Dirac operator* is defined as

$$\mathcal{D} = i^{-1} \sum_{k=1}^n \alpha_k \frac{\partial}{\partial x_k} = i^{-1}(\alpha \cdot \nabla),$$

and the 4×4 Dirac matrices can be written as

$$(1.2) \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

in terms of the Pauli matrices

$$(1.3) \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The α matrices satisfy the following relations

$$\begin{aligned} \alpha_j \alpha_k + \alpha_k \alpha_j &= 2\delta_{jk} I_4, & 1 \leq j, k \leq 3, \\ \alpha_j \beta + \beta \alpha_j &= 0, & j = 1, 2, 3, \\ \beta^2 &= I_4; \end{aligned}$$

as a consequence, the following identity holds

$$(1.4) \quad (i\partial_t - \mathcal{D} - m\beta)(i\partial_t + \mathcal{D} + m\beta)u = (\Delta - m^2 - \partial_{tt}^2)\mathbb{I}_4 u.$$

This identity allows us to study the free Dirac equation through a system of decoupled Klein-Gordon (or wave, in the mass-less case) equations. Therefore, it is not a difficult task to deduce dispersive estimates (time-decay, Strichartz...) for the Dirac flow from the corresponding ones of their more celebrated Klein-Gordon

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or wave counterparts. Moreover, this fact stresses the substantial difference between the mass-less and the massive cases in equation (1.1). Of course, when perturbations terms appear in equation (1.1), as potentials or nonlinear terms, the argument above needs to be handled with a lot of additional care, and in particular is going to fail in low regularity settings, when the structures of the single terms play crucial roles. The study of dispersive estimates for the Dirac equation with potentials has already been dealt with in literature: we mention at least the papers [10], [11], [4], [7], [6] in which various several of estimates are discussed for electric and magnetic perturbations of equation (1.1).

In the last years, a lot of effort has been spent in order to investigate higher order perturbations of dispersive partial differential equations: in particular, the problem of understanding how variable coefficients perturbations affect the wave and Schrödinger flows has attracted growing interest in the community. The interest of this kind of problems is of geometric nature, as it is indeed natural to interpret the variable coefficients as a "change of metric", and therefore to recast the problem as the study of dispersive dynamics on non-flat manifolds. It turns out that in this contest a crucial role is played by the so called *non-trapping condition* on the coefficients that, roughly speaking, is a condition that prevents geodesic flows to be confined in compact sets for large times: the failure of such a condition is indeed understood to be an obstacle to dispersion. Such a condition is in fact guaranteed in case of "small perturbations" of the flat metric. On the subject of dispersion for Klein-Gordon and wave equations, we mention, but this is not exhaustive, [1, 2, 3, 14, 18, 19, 20].

The aim of this manuscript is to provide some first results in this framework for the Dirac equation for which, to the best of our knowledge, nothing is known in this contest; in particular we here aim to prove weak dispersive estimates for its flow under some different assumptions on the geometry. We stress the fact that, due to the rich algebraic structure of the Dirac operator, its generalization to curved spaces is significantly more delicate than the one of the Laplacian; we dedicate section 3 to this issue. On the other hand, once the equation is correctly settled, it is possible to rely on the the squaring trick as in the free case to reduce to a suitable variable coefficients wave equation with a lower order term, for which the full power of the multiplier technique can be exploited.

The general form of the Dirac operator on a manifold with a given metric $g_{\mu\nu}$ is the following

$$(1.5) \quad \mathcal{D} = i\gamma^a e_a^\mu D_\mu$$

where the matrices $\gamma_0 = \beta$ and $\gamma^j = \gamma^0 \alpha_j$ for $j = 1, 2, 3$, e^μ is a *vierbein* and D_μ defines the covariant derivative for fermionic fields (we postpone to forthcoming section 3 all the details and definitions). In what follows, we shall restrict to

metrics $g_{\mu\nu}$ that have the following structure

$$(1.6) \quad g_{\mu\nu} = \begin{cases} \phi^{-2}(t) & \text{if } \mu = \nu = 0 \\ 0 & \text{if } \mu\nu = 0 \text{ and } \mu \neq \nu \\ -h_{\mu\nu}(\vec{x}) & \text{otherwise.} \end{cases}$$

In other words, time and space are decoupled. What is more, we assume that the manifold is complete. This ensures that the Dirac operator is self-adjoint, see [9], a property we use for the conservation of energy or for estimates of norms in terms of this operator. The same assumption is made in [14].

The function ϕ is assumed to be strictly positive for all t . Let us remark that with a change of variable on time, one may take ϕ equal to 1.

Within this setting, we will show that equation $(\mathcal{D} + m)u = 0$ can be written in the more convenient form

$$(1.7) \quad i\phi\partial_t u - Hu = 0$$

where H is an operator such that $H^2 = -\Delta_h + \frac{1}{4}\mathcal{R}_h + m^2$, and Δ_h and \mathcal{R}_h are respectively the Laplace-Beltrami operator and the scalar curvature associated to the spatial metrics h . As a consequence, it can be proved that if u solves equation (1.7) then u solves the equation

$$(1.8) \quad -(\phi\partial_t)^2 u + \Delta_h u - \frac{1}{4}\mathcal{R}_h u - m^2 u = 0.$$

We should stress the analogy with the free case; the scalar curvature term that appears in the equation above vanishes when reducing to the Minkowski metrics.

Our first main result concerns the asymptotically flat case (we postpone all the details and the assumptions to forthcoming subsection 2.2).

Theorem 1.1. *Let u be a solution to (1.7) with initial condition u_0 with g satisfying (1.6). Assuming that h is "asymptotically flat enough", then let $\eta_1, \eta_2 > 0$, there exists $C_{\eta_1, \eta_2} > 0$ independent from u such that*

$$(1.9) \quad \|\langle x \rangle^{-3/2-\eta_1} u\|_{L_\phi^2 L_x^2} + \|\langle x \rangle^{-1/2-\eta_2} \nabla u\|_{L_\phi^2 L_x^2} \leq C_{\eta_1, \eta_2} \|Hu_0\|_{L_{M_h}^2}.$$

The norm $\|\cdot\|_{L^2(M_h)}$ is the L^2 norm on the manifold M_h , that is

$$\|f\|_{L^2(M_h)}^2 = \int_{D(h)} |f(x)|^2 \sqrt{\det(h(x))} dx$$

where $D(h)$ is the set where h is defined. In the asymptotically flat case, this set is assumed to be \mathbb{R}^3 .

Remark 1.1. By "asymptotically flat enough" we mean that h satisfies the assumptions listed in Subsection 2.2, i.e. (2.6)-(2.11), with the constants involved small enough.

Remark 1.2. In fact, we can prove under assumptions of Theorem 1.1 a slightly stronger version of estimate (1.9), namely the following

$$(1.10) \quad \|u\|_{XL_\phi^2}^2 + \|\nabla u\|_{YL_\phi^2}^2 \leq C_{\nu, N, \sigma} \|Hu_0\|_{L_{M_h}^2}^2$$

where the Campanato-type norms X and Y are defined in Subsection 2.1, by the equations (2.3), (2.4). These spaces represent indeed somehow the "natural" setting when dealing with the multiplier method (see e.g. [4], [8]); nevertheless, we prefer to state our Theorem in this form for the sake of symmetry with the next result. We stress anyway that estimate (1.10), which is the one that we will prove, implies (1.9).

Remark 1.3. As done in [8] for the Helmholtz equation, our proof allows us, after carefully following all the constants produced by the various estimates, to provide some explicit sufficient conditions that guarantee local smoothing: we indeed quantify the closeness to a flat metric which we require by giving out explicit inequalities that the constants in Assumptions (2.9)-(2.10) must satisfy to get the result. This fact, as mentioned, is strictly connected to the geometrical assumption of *non-trapping* of the metric g_{jk} ; therefore our strategy of proof gives, in a way, some explicit sufficient conditions that ensure the metric g to be non-trapping.

Remark 1.4. In Minkowski space-time, the influence of a magnetic potential in equation (1.1) is reflected in the change of the covariant derivative, that is the substitution

$$\nabla \rightarrow \nabla_A := \nabla - iA$$

where $A = A(x) = (A^1(x), A^2(x), A^3(x)) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the magnetic potential. This phenomenon can be generalized to equation (1.7): the presence of a magnetic potential has indeed essentially the effect of changing the covariant derivative D_μ . In particular, by squaring the magnetic Dirac equation on a space with a metric g_{jk} with the structure (1.6) one obtains the following Klein-Gordon type equation

$$-(\phi \partial_t)^2 u + \tilde{\Delta}_h u - \frac{1}{4} \mathcal{R}_h u - m^2 u - \frac{1}{2} F_{jk} [\gamma_i, \gamma_k] u = 0.$$

where $\tilde{\Delta}_h$ is the magnetic Laplace-Beltrami operator and $F_{jk} = \partial_j A_k - \partial_k A_j$ is the electromagnetic field tensor. The strategy of the present paper allows to deal with this more general situation: anyway, we prefer not to include magnetic potentials in order to keep our presentation more readable. We refer the interested reader to [8], in which the electromagnetic Helmholtz equation is discussed with the same techniques as here.

Remark 1.5. The problem of proving Strichartz estimates for solutions to equation (1.7) seems significantly more difficult: variable coefficients perturbations indeed prevent the use of the standard Duhamel trick to handle the additional terms (see e.g. [4]) and requires a completely different approach involving phase space analysis and parametrics construction, (see e.g. [15] for the wave equation). We

intend to deal with this problem in forthcoming papers. We also mention the fact that, by mimicking the author's argument in [5], one could prove global (in time) Strichartz estimates for solutions to equation (1.8) (and thus to (1.7)) from local smoothing estimates (1.9) provided the metric h_{jk} is also assumed to be flat outside some ball and the solutions to (1.8) which are compactly supported in space are known to satisfy local (in time) Strichartz estimates. This would give, at least, a conditional result.

Next, we consider the specific case of the so called *wrapped products*, that is metrics of the form (1.6) with the additional structure

$$(1.11) \quad h_{11} = 1, h_{1i} = h_{i1} = 0 \text{ if } i \neq 1, h_{ij} = d(x^1)\beta_{ij}(x^2, x^3)$$

where β is a 2×2 metrics. We denote the scalar curvature of β by $\mathcal{R}_\beta = \mathcal{R}_\beta(x^2, x^3)$.

We will use the more comfortable (and intuitive) notation $r = x^1$. In the case of the flat metrics of \mathbb{R}^3 , $d(r) = r^2$ and β is the metrics of the sphere S^2 .

We prove the following theorem.

Theorem 1.2. *Let u be a solution to (1.7) with initial condition u_0 , with g satisfying (1.6) and h as in (1.11). Then the following results hold.*

- (Hyperbolic-type metrics) Take $d(r) = e^{r/2}$ in (1.11) and assume that for all (x^2, x^3) , $\mathcal{R}_\beta(x^2, x^3) \geq 0$ and $m^2 > \frac{3}{32}$. Let $\eta_1, \eta_2 > 0$, there exists $C_{\eta_1, \eta_2} > 0$ such that for all u solution of the linear Dirac equation, we have

$$(1.12) \quad \|\langle r \rangle^{-(1+\eta_1)} u\|_{L^2(M_g)}^2 + \|\langle r \rangle^{-(1/2+\eta_2)} \nabla_h u\|_{L^2(M_g)}^2 \leq C_{\eta_1, \eta_2} \|Hu_0\|_{L^2(M_h)}.$$

- (Flat-type metrics) Take $d(r) = r^2$ in (1.11) and assume that for all (x^2, x^3) , $\mathcal{R}_\beta \geq 2$. Let $\eta_1, \eta_2 > 0$, there exists $C_{\eta_1, \eta_2} > 0$ such that for all u solution of the linear Dirac equation, we have

$$(1.13) \quad \|\langle r \rangle^{-(3/2+\eta_1)} u\|_{L^2(M_g)}^2 + \|\langle r \rangle^{-(1/2+\eta_2)} \nabla_h u\|_{L^2(M_g)}^2 \leq C_{\eta_1, \eta_2} \|Hu_0\|_{L^2(M_h)}.$$

- (Sub-flat type metrics) Take $d(r) = r^n$ in (1.11) with $n \in]2 - \sqrt{2}, 4/3]$. There exists $C_n > 0$ such that if for all (x^2, x^3) , $\mathcal{R}_\beta \geq C_n$, then for all $\eta_1, \eta_2 > 0$, there exists $C_{\eta_1, \eta_2, n} > 0$ such that for all u solution of the linear Dirac equation, we have

$$(1.14) \quad \|\langle r \rangle^{-(3/2+\eta_1)} u\|_{L^2(M_g)}^2 + \|\langle r \rangle^{-(1/2+\eta_2)} \nabla_h u\|_{L^2(M_g)}^2 \leq C_{\eta_1, \eta_2, n} \|Hu_0\|_{L^2(M_h)}.$$

The norm $\|\cdot\|_{L^2(M_g)}$ is the L^2 norm on the manifold M_g , that is

$$\|f\|_{L^2(M_g)}^2 = \int_{\mathbb{R} \times D(h)} |f(t, x)|^2 \sqrt{\det(g(t, x))} dx dt$$

where $D(h)$ is the set where h is defined and with $\nabla_h f \cdot \nabla_h$ we are denoting the operator $h^{ij} \partial_i f \partial_j$. With the structure of g , we get $g(t, x) = -\phi^{-2}(t)h(x)$, which yields

$$\|f\|_{L^2(M_g)}^2 = \int_{\mathbb{R}} \|f(t, \cdot)\|_{L^2(M_h)}^2 \phi^{-1}(t) dt.$$

The strategy for proving these results relies on the multiplier method: using some standard integration by parts techniques we will be able to build a proper virial identity for equation (1.7) (see Proposition 4.2) which, by choosing suitable multiplier functions, will allow us to prove local smoothing estimates.

Remark 1.6. By mixing the techniques used to prove Theorem 1.1 and Theorem 1.2, one should expect to be able to prove local smoothing for the Dirac equation on metrics that are asymptotically like the wrapped products we presented.

Remark 1.7. One should also be able, simply by mimicking techniques, to prove the same theorem for a more generic wrapped product. The cases we presented include a hyperbolic type metrics and a flat-like metrics for which the geometry, represented as an extra term in the virial identity, is neither a help nor an obstacle. For the sub-flat metrics though, the geometry is an help. Keeping in mind these two different mechanisms, one should be able to design a more general setting.

The plan of the paper is the following: in section 2 we build our setup, fixing the notations and providing some preliminary inequalities that will be needed in the following, in section 3 we review the theory of Dirac operators on curved spaces, showing how to properly build a dynamical equation, in section 4 we prove the virial identity that is the crucial stepping stone for local smoothing with the use of the multiplier method, while sections 5 and 6 are devoted, respectively, to the proofs of Theorems 1.1 and 1.2.

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2. PRELIMINARIES

We dedicate this section to fix notations and expose some preliminary useful results.

2.1. Notations. We start by recalling some classical notations from differential geometry that we will use in the rest of the paper. Let $h = h(x)$ be a 3×3 , positive definite, real matrix that defines, in a standard way, a metric tensor. We recall that the *scalar curvature* can be written as

$$(2.1) \quad \mathcal{R}_h = h^{jk} \left(\frac{\partial}{\partial x_i} \Gamma_{jk}^i - \frac{\partial}{\partial x_k} \Gamma_{ji}^i + \Gamma_{jk}^\ell \Gamma_{i\ell}^i + \Gamma_{ji}^\ell \Gamma_{k\ell}^i \right).$$

where Γ_{jk}^i denote the standard Christoffel symbols. In what follows we will use the compact notation for the matrices

$$h = h(x) = [h_{jk}(x)]_{j,k=1}^3 \quad h_{inv} = h_{inv}(x) = [h^{jk}(x)]_{j,k=1}^3.$$

We will need the quantities

$$\hat{h}(x) = h^{jk} \hat{x}_j \hat{x}_k, \quad \bar{h}(x) = \text{Tr}(h_{inv}(x)) = h^{kk}(x)$$

where we are using the standard conventions for implicit summation and $\hat{x} = x/|x|$. Notice that, as $h(x)$ is assumed to be positive definite,

$$0 \leq \hat{h}(x) \leq \bar{h}(x)$$

for every x . Also, we will use the compact notation

$$\tilde{h}^{jk} = \sqrt{\det(h)} h^{jk}.$$

Straightforward computations show that, for every sufficiently regular radial function ψ ,

$$(2.2) \quad \Delta_h \psi(x) = \hat{h} \psi'' + \frac{\bar{h} - \hat{h}}{|x|} \psi' + \frac{1}{\sqrt{\det(h)}} \partial_j (\tilde{h}^{jk}) \hat{x}_k \psi'$$

where $'$ denotes the radial derivative and we are slightly abusing notations by identifying the functions $\psi(x)$ and $\psi(|x|)$.

The natural scalar product induced by h is given by

$$\langle f, g \rangle_h = \int_{M_h} f g = \int_{\mathbb{R}^3} \overline{f(x)} g(x) \sqrt{\det(h)} d^3 x.$$

The corresponding $\|\cdot\|_{L^2(M_h)}$ norm is naturally defined.

We now introduce the functional spaces that will represent the setting for our estimates in the asymptotically flat case. We define the *Campanato-type norms* (note that $\langle R \rangle = \sqrt{1 + R^2}$) as

$$(2.3) \quad \|v\|_X^2 := \sup_{R>0} \frac{1}{\langle R \rangle^2} \int_{M_h \cap S_R} |v|^2 dS = \sup_{R>0} \frac{1}{\langle R \rangle^2} \int_{S_R} |v|^2 \sqrt{\det(h)} dS$$

where dS denotes the surface measure on the sphere $\{|x| = R\}$, and

$$(2.4) \quad \|v\|_Y^2 := \sup_{R>0} \frac{1}{\langle R \rangle} \int_{M_h \cap B_R} |v|^2 dx = \sup_{R>0} \frac{1}{\langle R \rangle} \int_{B_R} |v|^2 \sqrt{\det(h)} dx$$

where we are denoting with S_R and B_R , respectively, the surface and the interior of the sphere of radius R centred in the origin. Notice that

$$(2.5) \quad \|v\|_Y^2 \sim \sup_{R \geq 1} \frac{1}{R} \int_{M_h \cap B_R} |v|^2,$$

The norm with respect to time will be given by

$$\|u\|_{L_{\phi,T}^2}^2 = \int_0^T \frac{|u(t)|^2}{\phi(t)} dt, \quad \|u\|_{L_\phi^2}^2 = \int_0^{+\infty} \frac{|u(t)|^2}{\phi(t)} dt$$

where ϕ is the positive function given in the definition of g . In particular, when $\phi = 1$ these norms recover the standard L_T^2 (resp. L^2) ones, and we will simply denote with $L_T^2 = L_{1,T}^2$.

We mean by \int_{M_h} the integral over the manifold with the volume unit of the manifold, that is

$$\int_{M_h} f = \int_{D(h)} f(x) \sqrt{\det(h(x))} dx$$

where $D(h)$ is the set where h is defined.

Finally, we recall that by $\nabla_h f \cdot \nabla_h$ we mean the operator $h^{ij} \partial_i f \partial_j$.

2.2. Assumptions for Theorem 1.1. We here collect the assumptions we make on the matrix $h(x) = [h_{jk}(x)]_{j,k=1}^n$ for Theorem 1.1. First of all, we require the following natural matrix-type bounds to hold for every $x, \xi \in \mathbb{R}^3$

$$(2.6) \quad \nu |\xi|^2 \leq h^{jk}(x) \xi_j \xi_k \leq N |\xi|^2$$

which is equivalent to

$$(2.7) \quad N^{-1} |\xi|^2 \leq h_{jk}(x) \xi_j \xi_k \leq \nu^{-1} |\xi|^2.$$

A consequence of (2.6) is that

$$|h_{inv} v|^2 \cong |h v|^2 \cong_{\nu, N} |v|^2.$$

Moreover, (2.6) implies

$$(2.8) \quad N^{-3/2} \leq \sqrt{\det(h(x))} \leq \nu^{-3/2}, \quad \forall x \in \mathbb{R}^3$$

where $\det(h(x)) = \det[h_{jk}(x)]_{j,k=1}^n$. Then, we impose an asymptotically flatness condition in the form

$$(2.9) \quad |h_{inv}(x) - I| \leq C_I \langle x \rangle^{-\sigma}$$

and

$$(2.10) \quad |h'_{inv}(x)| + |x| |h''_{inv}(x)| + |x|^2 |h'''_{inv}(x)| \leq C_h \langle x \rangle^{-1-\sigma}, \quad \sigma \in (0, 1),$$

where we are denoting with $|h(x)|$ the operator norm of the matrix $h(x)$ and with $|h'| = \sum_{|\alpha|=1} |\partial^\alpha h(x)|$, $|h''| = \sum_{|\alpha|=2} |\partial^\alpha h(x)|$ and $|h'''| = \sum_{|\alpha|=3} |\partial^\alpha h(x)|$.

Note also that these assumptions imply

$$(2.11) \quad \|\langle x \rangle^{1+\alpha} \nabla(\sqrt{\det(h(x))})\|_{L^\infty} + \|\langle x \rangle^{1+\alpha} \nabla(\sqrt{\det(h(x))} h^{jk}(x))\|_{L^\infty} \leq C_\nabla$$

for some $\alpha = \alpha(\sigma) \in (0, 1)$ and some constant C_∇ (the constant C_∇ might be explicitly written in terms of C_h , ν and N , but here we prefer to introduce another constant to keep notations lighter).

Remark 2.1. Conditions listed above are fairly standard in this setting (see [8] and references therein), and are to be thought of as defining asymptotically flat metrics. The main example we have in mind is given by the choice $h_{jk} = (1 + \varepsilon \langle x \rangle^{-\sigma}) \delta_{jk}$ for some ε sufficiently small: this matrix satisfies indeed all the assumptions of this subsection. As a further particular case we can think h_{jk} to be a small and regular enough compactly supported perturbation of the flat metrics.

2.3. Useful inequalities. The following Hardy-type estimate will be used several times throughout the paper.

Proposition 2.1. *Let $m \geq 0$. For any f such that $Hf \in L^2_{M_h}$ the following inequality holds*

$$(2.12) \quad m^2 \int_{M_h} |f|^2 + C_H^{-1} \int_{M_h} \frac{|f|^2}{|x|^2} \leq \int_{M_h} |Hf|^2$$

where the constant C_H^{-1} is given by

$$C_H^{-1} = \frac{\nu^4}{4} - C(\sigma)$$

with ν , N is as in (2.6) and $C(\sigma)$ is such that $|\mathcal{R}_h| \leq \frac{C(\sigma)}{\langle x \rangle^{(1+\sigma')|x|}}$ for some $\sigma' \in (0, 1)$.

Proof. We assumed that g was complete, such that the Dirac operator $\gamma^0 \mathcal{D}$ is self-adjoint. We refer to [9]. Since $\gamma^0 \mathcal{D} = i\phi \partial_t - H - \gamma^0 m$, we get that H is self-adjoint.

We write, as the operator H is self-adjoint with respect to the inner product defined by h ,

$$(2.13) \quad \int_{M_h} |Hf|^2 = -\langle H^2 f, f \rangle_{M_h} = \int_{M_h} (m^2 - \Delta_h) f f + \frac{1}{4} \int_{M_h} \mathcal{R}_h |f|^2 = I + II.$$

Notice now that

$$(2.14) \quad I = \int_{M_h} \nabla_h f \cdot \nabla_h f + m^2 \int_{M_h} |f|^2$$

Notice that assumption (2.6) implies

$$\int_{M_h} \nabla_h f \cdot \nabla_h f \geq \nu^{5/2} \int_{\mathbb{R}^3} |\nabla f|^2$$

which, by the application of standard Hardy's inequality

$$\int_{\mathbb{R}^3} \frac{|f|^2}{|x|^2} \leq 4 \int_{\mathbb{R}^3} |\nabla f|^2$$

can be estimated with

$$I \geq \frac{\nu^4}{4} \int_{M_h} \frac{|f|^2}{|x|^2}.$$

Moreover, our assumptions on h (2.10) imply

$$|\mathcal{R}_h| \leq \frac{C(\sigma)}{\langle x \rangle^{(1+\sigma')} |x|},$$

and thus

$$(2.15) \quad II \leq C(\sigma) \int_{M_h} \frac{|f|^2}{\langle x \rangle^{(1+\sigma')} |x|} \leq C(\sigma) \int_{M_h} \frac{|f|^2}{|x|^2}$$

where the constant $C(\sigma)$ is small. Putting all together, we thus have

$$\int_{M_h} |Hf|^2 \geq \left[\frac{\nu^4}{4} - C(\sigma) \right] \int_{M_h} \frac{|f|^2}{|x|^2}$$

which concludes the proof. \square

Remark 2.2. The following slightly more general inequality, which holds for any $\varepsilon \in (0, 1)$, will be useful in the paper:

$$(2.16) \quad m^2 \int_{M_h} |f|^2 + \left[\frac{1-\varepsilon}{4} \nu^4 - C(\sigma) \right] \int_{M_h} \frac{|f|^2}{|x|^2} + \varepsilon \int_{M_h} |\nabla f|^2 \leq \int_{M_h} |Hf|^2$$

which can be obtained by combining (2.12) with the obvious inequality

$$\varepsilon \int_{M_h} |\nabla f|^2 + \frac{(1-\varepsilon)}{4} \nu^4 \int_{M_h} \frac{|f|^2}{|x|^2} \leq \int_{M_h} |\nabla f|^2.$$

In what follows, we will also make use of a number of weighted inequalities that we collect in the following

Proposition 2.2. *For any $\sigma \in (0, 1)$ and any $v \in C_0^\infty(\mathbb{R}^n)$ the following estimates hold*

$$(2.17) \quad \int_{M_h} \frac{|u|^2}{\langle x \rangle^{1+\sigma}} \leq 8\sigma^{-1} C_{N,\nu} \|u\|_Y^2,$$

$$(2.18) \quad \sup_{R>1} \int_{M_h \cap B_R^c} \frac{R^2}{|x|^5} |u|^2 \leq C_{N,\nu} \|u\|_X^2,$$

$$(2.19) \quad \int_{M_h \cap B_1^c} \frac{|u|^2}{|x|^2 \langle x \rangle^{1+\sigma}} \leq 2\sigma^{-1} C_{N,\nu} \|u\|_X^2.$$

$$(2.20) \quad \|u\|_X^2 \leq C_{N,\nu} \left[4 \sup_{R>1} \frac{1}{R^2} \int_{M_h \cap S_R} |u|^2 + 13 \|\nabla u\|_Y^2 \right]$$

$$(2.21) \quad \|u\|_Y^2 \leq 3C_{N,\nu} (2\|\nabla u\|_Y^2 + \|u\|_X^2)$$

$$(2.22) \quad \int_{M_h} \frac{|u|^2}{|x|^2} \leq C_{N,\nu} \int_{M_h} |\nabla u|^2,$$

where the constant $C_{N,\nu} = \left(\frac{\nu}{N}\right)^{3/2}$.

Proof. For the proof we refer to [8] Section 3: the generalization from the Euclidean case to our perturbative setting is straightforward under assumptions (2.6)-(2.8). \square

3. FROM VIERBEIN TO DREIBEIN

Our aim in this subsection is to describe the Dirac equation on curved space-time in the case of a metric which dissociate time and space. We prove that it can be written

$$i\phi\partial_t u - Hu = 0$$

with ϕ a function appearing in the metric and H an operator such that

$$H^2 = -\Delta_h + \frac{1}{4}\mathcal{R}_h + m^2$$

where h is the space metric, Δ_h the Laplace-Beltrami operator associated to this metric, $m \in \mathbb{R}^+$ is a parameter (a mass) and \mathcal{R}_h the scalar curvature.

3.1. Vierbein. Let us recall which form takes the Dirac equation on curved space (of dimension 1+3).

Let $g_{\mu\nu}$ be a metric.

A vierbein associated to this metric is a matrix e_a^μ such that

$$g^{\mu\nu} = e_a^\mu \eta^{ab} e_b^\nu$$

where a, b, μ and ν are taken in $\{0, 1, 2, 3\}$ and with the conventional notation

$$\eta^{ab} = \begin{cases} 1 & \text{if } a = b = 0 \\ -1 & \text{if } a = b \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The vierbein is what makes the connection between the metric g and the Minkowski flat space-time.

The spin connection is defined as :

$$\omega_\mu^{ab} = e_\nu^a \partial_\mu e^{\nu b} + e_\nu^a \Gamma_{\mu\sigma}^\nu e_b^\sigma$$

where we recall that Γ is the Christoffel symbol or affine connection given by

$$\Gamma_{\mu\sigma}^\nu = \frac{1}{2}g^{\nu\lambda}(\partial_\mu g_{\lambda\sigma} + \partial_\sigma g_{\mu\lambda} - \partial_\lambda g_{\mu\sigma}).$$

We recall that in a flat space-time the (mass-less) Dirac operator is given by

$$\mathcal{D} = i\gamma^a \partial_a$$

where the γ^a are the usual Dirac matrices :

$$(3.1) \quad \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Note that $\gamma^0 = \beta$ and $\gamma^i = \gamma^0 \alpha_i$ for $i \in \{1, 2, 3\}$.

The linear Dirac equation is given by :

$$(\mathcal{D} + m)u = 0.$$

In the case of a curved space time ∂_a is replaced by $e_a^\mu D_\mu$ where D_μ is the covariant derivative for fermionic fields and is given by

$$D_\mu = \partial_\mu + \frac{1}{8} \omega_\mu^{ab} [\gamma_a, \gamma_b]$$

where $[\cdot, \cdot]$ is the commutator. We refer to [16], pages 221 to 229. The Dirac operator is then given by

$$\mathcal{D} = i\gamma^a e_a^\mu D_\mu.$$

One important fact about it is that

$$\mathcal{D}^2 = -\square_g - \frac{1}{4} \mathcal{R}_g$$

where \mathcal{R}_g is the scalar curvature associated to the metric g and \square_g is the d'Alembertian associated to the metrics g , that is, $\square_g = (\phi \partial_0)^2 - \Delta_h$.

The idea behind this formalism is that the vierbein and the spin connection are the link between curved space g and the flat space η . It is explained through Gauge theory and is detailed in [16].

3.2. Dreibein. The title dreibein is an abuse of language as what we call dreibein is not a vierbein in dimension 1+2 but the link between a metric and a Euclidean flat space of dimension 3.

We consider a metric g of the following form

$$g_{\mu\nu} = \begin{cases} \phi^{-2}(x^0) & \text{if } \mu = \nu = 0 \\ 0 & \text{if } \mu\nu = 0 \text{ and } \mu \neq \nu \\ -h_{\mu\nu}(\vec{x}) & \text{otherwise.} \end{cases}$$

where $\vec{x} = (x^1, x^2, x^3)$.

Let f_a^i be a so-called dreibein hence satisfying

$$h^{ij} = f_a^i f_a^j.$$

In this sum, a is taken only between 1 and 3. Note that we can and do choose f independent from x^0 .

Proposition 3.1. *Write*

$$e_a^\mu = \begin{cases} \phi(x^0) & \text{if } \mu = a = 0 \\ 0 & \text{if } \mu a = 0 \text{ and } \mu \neq a \\ f_a^\mu & \text{otherwise.} \end{cases}$$

The matrix e_a^μ is a vierbein for g .

Proof. We have

$$e_a^0 \eta^{ab} e_b^0 = \phi^2 \delta_a^0 \eta^{ab} \delta_b^0 = \phi^2 \eta^{00} = g^{00}$$

where δ_a^b is the Kronecker symbol.

What is more, with $i \neq 0$,

$$e_a^0 \eta^{ab} e_b^i = \phi e_{00}^i = 0 = g^{0i}$$

and for the same reason $e_a^i \eta^{ab} e_b^0 = g^{i0}$.

And finally, with $ij \neq 0$,

$$e_a^i \eta^{ab} e_b^j = f_a^i \eta^{ab} f_b^j = f_a^i (-\delta^{ab}) f_b^j = -f_a^i f_a^j = -h^{ij} = g^{ij}.$$

This makes e_a^μ a suitable vierbein for g . □

Let us see how the Christoffel symbol is changed.

Proposition 3.2. *Let*

$$\Lambda_{ij}^k = \frac{1}{2} h^{kl} (\partial_i h_{lj} + \partial_j h_{il} - \partial_l h_{ij}).$$

We have

$$\Gamma_{\mu\nu}^\sigma = \begin{cases} -\phi^{-1} \phi' & \text{if } \mu = \nu = \sigma = 0 \\ \Lambda_{\mu\nu}^\sigma & \text{if } \mu\nu\sigma \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\Gamma_{\mu\nu}^0 = \frac{1}{2} g^{0\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}).$$

Since $g^{0\lambda} = 0$ if $\lambda \neq 0$, we get

$$\Gamma_{\mu\nu}^0 = \frac{1}{2} \phi^2 (\partial_\mu g_{0\nu} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu}).$$

Because $g_{0\nu} = g_{\nu 0} = 0$ if $\nu \neq 0$ and g_{00} depends only on x^0 , and h does not depend on x^0 , we have

$$\partial_\mu g_{0\nu} = \partial_\nu g_{\mu 0} = \partial_0 g_{\mu\nu} = 0$$

if μ or ν is not equal to 0. This yields

$$\Gamma_{\mu\nu}^0 = 0$$

if $\mu + \nu \neq 0$ and $\Gamma_{00}^0 = -\phi^{-1} \phi'$.

What is more,

$$\Gamma_{0\nu}^\sigma = \frac{1}{2}g^{\sigma\lambda}(\partial_0 g_{\lambda\nu} + \partial_\nu g_{0\lambda} - \partial_\lambda g_{0\nu}).$$

If $\nu \neq 0$, we have $\partial_0 g_{\lambda\nu} = \partial_\nu g_{0\lambda} = \partial_\lambda g_{0\nu} = 0$. If $\sigma \neq 0$, then the sum over λ is restricted to $\lambda \neq 0$ and hence $\partial_0 g_{\lambda\nu} = \partial_\nu g_{0\lambda} = \partial_\lambda g_{0\nu} = 0$. Hence,

$$\Gamma_{0\nu}^\sigma = 0$$

if $\nu + \gamma \neq 0$. For symmetry reasons, $\Gamma_{\mu 0}^\sigma = 0$ if $\mu + \sigma \neq 0$.

Finally, we have

$$\Gamma_{\mu\nu}^\sigma = \begin{cases} -\phi^{-1}\phi' & \text{if } \mu = \nu = \sigma = 0 \\ \Lambda_{\mu\nu}^\sigma & \text{if } \mu\nu\sigma \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

□

Let us see how the spin connection is changed.

Proposition 3.3. *Let*

$$\alpha_i^{ab} = f_j^a \partial_i f^{jb} + f_j^a \Lambda_{ik}^j f^{kb}.$$

We have

$$\omega_\mu^{ab} = \begin{cases} 2\phi'\phi^3 & \text{if } \mu = a = b = 0 \\ \alpha_\mu^{ab} & \text{if } \mu\nu\sigma \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

Proof. Indeed, we have

$$\omega_0^{ab} = e_\nu^a \partial_0 e^{\nu b} + e_\nu^a \Gamma_{0\sigma}^\nu e^{\sigma b}.$$

Because $\Gamma_{0\sigma}^\nu = 0$ if $\sigma + \nu \neq 0$ we have $e_\nu^a \Gamma_{0\sigma}^\nu e^{\sigma b} = e_0^a \Gamma_{00}^0 e^{0b}$ which is equal to $-\phi^3\phi'$ if $a = b = 0$ and to 0 otherwise. Because $\partial_0 e^{\nu b} = 0$ if $\nu + b \neq 0$ and to $\partial_0 \phi^3 = 3\phi^2\phi'$ we get $e_\nu^a \partial_0 e^{\nu b} = 3\phi^3\phi'$ if $a = b = 0$ and to 0 otherwise. Therefore,

$$\omega_0^{ab} = 0$$

if a or b is different from 0 and $\omega_0^{00} = 2\phi'\phi^3$.

What is more,

$$\omega_\mu^{0b} = e_\nu^0 \partial_\mu e^{\nu b} + e_\nu^0 \Gamma_{\mu\sigma}^\nu e^{\sigma b} = \phi \partial_\mu e^{0b} + \phi \Gamma_{\mu\sigma}^0 e^{\sigma b} = \phi \partial_\mu e^{0b} + \phi \Gamma_{\mu 0}^0 e^{0b}.$$

If $b \neq 0$, then $e^{0b} = 0$ and hence $\omega_\mu^{0b} = 0$. If $\mu \neq 0$ then $\partial_\mu e^{0b} = 0$ and $\Gamma_{\mu 0}^0 = 0$ and hence $\omega_\mu^{0b} = 0$.

we also have

$$\omega_\mu^{a0} = e_\nu^a \partial_\mu e^{\nu 0} + e_\nu^a \Gamma_{\mu\sigma}^\nu e^{\sigma 0} = e_0^a \partial_\mu e^{00} + e_\nu^a \Gamma_{\mu 0}^\nu e^{00} = e_0^a \partial_\mu e^{00} + e_0^a \Gamma_{\mu 0}^0 e^{00}.$$

If $a \neq 0$ then $e_0^a = 0$ and hence $\omega_\mu^{a0} = 0$. If $\mu \neq 0$, then $\partial_\mu e^{00} = 0$ and $\Gamma_{\mu 0}^0 = 0$ and hence $\omega_\mu^{a0} = 0$.

Finally, we have

$$\omega_\mu^{ab} = \begin{cases} 2\phi'\phi^3 & \text{if } \mu = a = b = 0 \\ \alpha_\mu^{ab} & \text{if } \mu\nu\sigma \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

□

Covariant derivative and Dirac operator

The covariant derivative is given by

$$D_0 = \partial_0, \quad D_i = \partial_i + \frac{1}{8}\alpha_i^{ab}[\gamma_a, \gamma_b].$$

Therefore, the Dirac operator can be written

$$\mathcal{D} = i\gamma^0\phi\partial_0 + i\gamma^a f_a^j D_j.$$

Let

$$\mathcal{H} = i\gamma^a f_a^j D_j \text{ and } H = -\gamma^0(\mathcal{H} + m).$$

Proposition 3.4. *With these notations, we have*

$$(3.2) \quad H^2 = m^2 - \Delta_h + \frac{1}{4}\mathcal{R}_h.$$

Proof. First, we prove that $\mathcal{H}^2 = \Delta_h - \frac{1}{4}\mathcal{R}_h$.

We have

$$\mathcal{D}^2 = \mathcal{H}^2 + (i\gamma^0\phi\partial_0)^2 + (i\gamma^0\phi\partial_0\mathcal{H} + \mathcal{H}i\gamma^0\phi\partial_0).$$

Since γ^0 commutes with ϕ and ∂_0 and $(\gamma^0)^2 = 1$, we have $(i\gamma^0\phi\partial_0)^2 = -(\phi\partial_0)^2$.

Since ϕ and ∂_0 commute with \mathcal{H} and γ^0 , we have

$$(i\gamma^0\phi\partial_0\mathcal{H} + \mathcal{H}i\gamma^0\phi\partial_0) = i\phi\partial_0(\gamma^0\mathcal{H} + \mathcal{H}\gamma^0).$$

Given the Dirac matrices (3.1), we have for all $a > 0$, $\gamma^0\gamma_a = -\gamma_a\gamma^0$. Hence, γ^0 commutes with $[\gamma_a, \gamma_b]$ and thus with D_j . Therefore, we get

$$(\gamma^0\mathcal{H} + \mathcal{H}\gamma^0) = i(\gamma^0\gamma_a + \gamma_a\gamma^0)f_a^j D_j = 0.$$

We get

$$\mathcal{D}^2 = \mathcal{H}^2 - (\phi\partial_0)^2.$$

We recall that $\mathcal{D}^2 = -\square_g - \frac{1}{4}\mathcal{R}_g$ and given the metric $\square_g = (\phi\partial_0)^2 - \Delta_h$ and $\mathcal{R}_g = \mathcal{R}_h$. Therefore,

$$\mathcal{H}^2 = \Delta_h - \frac{1}{4}\mathcal{R}_h.$$

Finally, we have

$$H^2 = \gamma^0(\mathcal{H} + m)\gamma^0(\mathcal{H} + m)$$

and since m commutes with γ^0 and $(\gamma^0)^2 = Id$ we get

$$H^2 = (\gamma^0\mathcal{H}\gamma^0 + m)(\mathcal{H} + m)$$

and since \mathcal{H} anti-commutes with γ^0 and commutes with m , we get

$$H^2 = (-\mathcal{H} + m)(\mathcal{H} + m) = m^2 - \mathcal{H}^2 = m^2 - \Delta_h + \frac{1}{4}\mathcal{R}_h.$$

□

Besides

$$m + \mathcal{D} = \gamma^0(i\phi\partial_0 - H)$$

and

$$(i\phi\partial_0 + H)(i\phi\partial_0 - H) = -(\phi\partial_0)^2 - H^2 = -(\phi\partial_0)^2 + \Delta_h - \frac{1}{4}\mathcal{R}_h + m^2.$$

Remark 3.1. We shall use freely either ∂_0 or ∂_t for the time derivative.

Corollary 3.5. *If u solves the Dirac equation*

$$(3.3) \quad i\phi\partial_t u - Hu = 0$$

then u satisfies also

$$(3.4) \quad -(\phi\partial_t)^2 u + \Delta_h u - \frac{1}{4}\mathcal{R}_h u - m^2 u = 0.$$

Remark 3.2. With a change of variable, we can replace ϕ by $\phi = 1$, and we get the more simple expression,

$$m + \mathcal{D} = \gamma^0(i\partial_t - H),$$

and

$$(i\partial_t + H)(i\partial_t - H) = -\partial_t^2 + \Delta_h - \frac{1}{4}\mathcal{R}_h - m^2.$$

4. VIRIAL IDENTITY

We consider the linear equation

$$(4.1) \quad i\phi\partial_t u - Hu = 0.$$

We have that if u solves (4.1), then u also solves

$$(4.2) \quad (\phi\partial_t)^2 u + Lu = 0$$

with $L = H^2 = \frac{1}{4}\mathcal{R}_h + m^2 - \Delta_h$. Indeed, $(i\phi\partial_t + Hu)(i\phi\partial_t u - Hu) = -(\phi\partial_t)^2 u - H^2 u$.

Note that L is self-adjoint for the inner product $\langle \cdot, \cdot \rangle_h$. Let

$$\Theta(t) = \langle \psi\phi\partial_t u, \phi\partial_t u \rangle_h + \text{Re}\langle (2\psi L - L\psi)u, u \rangle_h$$

where ψ is a real valued function of space.

4.1. Differentiation. We compute $\phi\partial_t\Theta$ and $(\phi\partial_t)^2\Theta$ when u solves (4.1). The computation is the same as in the case of a flat metric and is mainly based on the self-adjointness of L .

Proposition 4.1. *We have that Θ satisfies*

$$(4.3) \quad \phi\partial_t\Theta = \operatorname{Re}\langle [L, \psi]u, \phi\partial_t u \rangle_h,$$

$$(4.4) \quad (\phi\partial_t)^2\Theta = -\frac{1}{2}\langle [L, [L, \psi]]u, u \rangle_h.$$

4.2. Commutators. We compute explicit formulae in terms of u and h of $\phi\partial_t\Theta$ and $(\phi\partial_t)^2\Theta$.

Proposition 4.2. *The explicit expressions of $\phi\partial_t\Theta$ and $(\phi\partial_t)^2\Theta$ in terms of u and h are*

$$(4.5) \quad \begin{aligned} \phi\partial_t\Theta &= -\operatorname{Re}\left(\int_{M_h} (\Delta_h\psi)\bar{u}\phi\partial_t u + 2\int_{M_h} \nabla_h\psi \cdot \nabla_h\bar{u}\phi\partial_t u\right) \\ (\phi\partial_t)^2\Theta &= \int_{M_h} \left(\frac{1}{2}(\Delta_h^2\psi) + \frac{1}{4}\nabla_h\psi \cdot \nabla_h\mathcal{R}_h\right)|u|^2 + 2\int_{M_h} (\partial_j\bar{u}\partial_i u)D^2(\psi)^{ij} \end{aligned}$$

where $D^2(\psi)^{ij} = h^{il}h^{kj}\partial_l\partial_k\psi - \Lambda^{k,ij}\partial_k\psi$, from which we deduce the virial identity

$$(4.6) \quad \begin{aligned} &-\int_{M_h} \left(\frac{1}{2}(\Delta_h^2\psi) + \frac{1}{4}\nabla_h\psi \cdot \nabla_h\mathcal{R}_h\right)|u|^2 + 2\int_{M_h} (\partial_j\bar{u}\partial_i u)D^2(\psi)^{ij} \\ &= -(\phi\partial_t)\operatorname{Re}\left(\int_{M_h} (\Delta_h\psi)\bar{u}\phi\partial_t u + 2\int_{M_h} \nabla_h\psi \cdot \nabla_h\bar{u}\phi\partial_t u\right). \end{aligned}$$

Proof. First, the commutator between $L = \frac{1}{4}\mathcal{R}_h + m^2 - \Delta_h$ and ψ is given by

$$[L, \psi] = [-\Delta_h, \psi] = -\Delta_h\psi - 2\nabla_h\psi \cdot \nabla_h$$

where $\nabla_h\psi \cdot \nabla_h$ is the operator given by

$$\nabla_h\psi \cdot \nabla_h\varphi = h^{ij}\partial_i\psi\partial_j\varphi.$$

We deduce from that

$$\phi\partial_t\Theta = -\operatorname{Re}\left(\int_{M_h} (\Delta_h\psi)\bar{u}\phi\partial_t u + 2\int_{M_h} \nabla_h\psi \cdot \nabla_h\bar{u}\phi\partial_t u\right).$$

We now have that, since m^2 commutes with everything,

$$[L, [L, \psi]] = [L, -\Delta_h\psi] + \left[\frac{1}{4}\mathcal{R}_h, -2\nabla_h\psi \cdot \nabla_h\right] + [\Delta_h, 2\nabla_h\psi \cdot \nabla_h] = M_1 + M_2 + M_3.$$

We have

$$M_1 = (\Delta_h)^2\psi + 2\nabla_h(\Delta_h\psi) \cdot \nabla_h, \quad M_2 = \frac{1}{2}\nabla_h\psi \cdot \nabla_h\mathcal{R}_h$$

and hence

$$\langle M_1 u, u \rangle_h = \int_{M_h} (\Delta_h)^2\psi|u|^2 + \int_{M_h} 2\nabla_h(\Delta_h\psi) \cdot \nabla_h\bar{u}u$$

and

$$\langle M_2 u, u \rangle_h = +\frac{1}{2} \int_{M_h} \nabla_h \psi \cdot \nabla_h \mathcal{R}_h |u|^2.$$

First, we compute $(2 \nabla_h \psi \cdot \nabla_h)^*$. We have for any test functions v, w

$$\begin{aligned} \langle 2 \nabla_h \psi \cdot \nabla_h v, w \rangle &= 2 \int \sqrt{\det(h)} h^{ij} \partial_i \psi \partial_j \bar{v} w d^3 x \\ &= -2 \int \partial_j (\sqrt{\det(h)} h^{ij} \partial_i \psi) \bar{v} w d^3 x - 2 \int \sqrt{\det(h)} h^{ij} \partial_i \psi \bar{v} \partial_j w \\ &= -2 \langle v, (\Delta_h \psi) w \rangle_h - 2 \langle v, \nabla_h \psi \cdot \nabla_h w \rangle_h \end{aligned}$$

in other words, $(2 \nabla_h \psi \cdot \nabla_h)^* = -2 \Delta_h \psi - 2 \nabla_h \psi \cdot \nabla_h$.

This gives in particular

$$\langle 2 \nabla_h (\Delta_h \psi) \cdot \nabla_h u, u \rangle = -2 \int_{M_h} (\Delta_h^2 \psi) |u|^2 - \langle u, 2 \nabla_h (\Delta_h \psi) \cdot \nabla_h u \rangle_h$$

and thus

$$(4.7) \quad \text{Re} \langle 2 \nabla_h (\Delta_h \psi) \cdot \nabla_h u, u \rangle = - \int_{M_h} (\Delta_h^2 \psi) |u|^2$$

which yields

$$\text{Re} \langle M_1 u, u \rangle_h = 0.$$

We deal with the M_3 term by directly taking the inner product.

We have

$$\langle M_3 u, u \rangle_h = \langle \Delta_h (2 \nabla_h \psi \cdot \nabla_h) u, u \rangle_h - \langle (2 \nabla_h \psi \cdot \nabla_h) \Delta_h u, u \rangle_h$$

and then, given the adjoint of $2 \nabla_h \psi \cdot \nabla_h$,

$$\langle (2 \nabla_h \psi \cdot \nabla_h) \Delta_h u, u \rangle_h = -\langle \Delta_h u, 2(\Delta_h \psi) u \rangle_h - \langle u, \Delta_h (2 \nabla_h \psi \cdot \nabla_h) u \rangle_h.$$

Therefore, we get

$$\text{Re} \langle M_3 u, u \rangle_h = 2 \text{Re} \langle \Delta_h (2 \nabla_h \psi \cdot \nabla_h) u, u \rangle_h + 2 \text{Re} \langle \Delta_h u, (\Delta_h \psi) u \rangle_h.$$

The second term is given by

$$\langle \Delta_h u, \Delta_h \psi u \rangle_h = - \int_{M_h} \nabla_h u \cdot \nabla_h (\Delta_h \psi u)$$

which decomposes into

$$\langle \Delta_h u, \Delta_h \psi u \rangle_h = -\langle \nabla_h (\Delta_h \psi) \cdot \nabla_h u, u \rangle - \int_{M_h} \Delta_h \psi \nabla_h u \cdot \nabla_h u$$

and thanks to previous computations, (4.7), we get

$$\text{Re} \langle \Delta_h u, \Delta_h \psi u \rangle_h = - \int_{M_h} \Delta_h \psi \nabla_h u \cdot \nabla_h u + \frac{1}{2} \int_{M_h} (\Delta_h^2 \psi) |u|^2.$$

By summing up, we get

$$\begin{aligned} \operatorname{Re}\langle [L, [L, \psi]]u, u \rangle &= \int_{M_h} \left((\Delta_h^2 \psi) + \frac{1}{2} \nabla_h \psi \cdot \nabla_h \mathcal{R}_h \right) |u|^2 \\ &\quad - 2 \int_{M_h} (\Delta_h \psi) \nabla_h u \cdot \nabla_h u + 2 \operatorname{Re}\langle \Delta_h (2 \nabla_h \psi \cdot \nabla_h) u, u \rangle_h. \end{aligned}$$

It remains to compute

$$\operatorname{Re} I = 2 \operatorname{Re}\langle \Delta_h (2 \nabla_h \psi \cdot \nabla_h) u, u \rangle_h.$$

We have

$$I = 4 \int \partial_i \left(\sqrt{\det(h)} h^{ij} \partial_j (h^{kl} \partial_k \psi \partial_l \bar{u}) \right) u d^3 x.$$

By integration by parts, we have

$$I = -4 \int \sqrt{\det(h)} h^{ij} \partial_j \left(h^{kl} \partial_k \psi \partial_l \bar{u} \right) \partial_i u d^3 x$$

which decomposes into

$$I = -4 \int \sqrt{\det(h)} h^{ij} \partial_j (h^{kl} \partial_k \psi) \partial_l \bar{u} \partial_i u d^3 x - 4 \int \sqrt{\det(h)} h^{ij} h^{kl} \partial_k \psi \partial_l \partial_j \bar{u} \partial_i u d^3 x.$$

Let II be the second term of the right hand side, that is,

$$II = -4 \int \sqrt{\det(h)} h^{ij} h^{kl} \partial_k \psi \partial_l \partial_j \bar{u} \partial_i u d^3 x.$$

By integration by parts, we have

$$II = 4 \int \partial_l \left(\sqrt{\det(h)} h^{ij} h^{kl} \partial_k \psi \partial_i u \right) \partial_j \bar{u} d^3 x$$

which decomposes into

$$II = 4 \int \partial_l \left(\sqrt{\det(h)} h^{ij} h^{kl} \partial_k \psi \right) \partial_i u \partial_j \bar{u} d^3 x - \overline{II}.$$

Hence,

$$\operatorname{Re} II = 2 \int \partial_l \left(\sqrt{\det(h)} h^{ij} h^{kl} \partial_k \psi \right) \partial_i u \partial_j \bar{u} d^3 x.$$

Therefore,

$$\begin{aligned} \operatorname{Re} I &= -4 \operatorname{Re} \int \sqrt{\det(h)} h^{ij} \partial_j (h^{kl} \partial_k \psi) \partial_l \bar{u} \partial_i u d^3 x \\ &\quad + 2 \int \partial_l \left(\sqrt{\det(h)} h^{ij} h^{kl} \partial_k \psi \right) \partial_i u \partial_j \bar{u} d^3 x \end{aligned}$$

that is

$$\operatorname{Re} I = \operatorname{Re} \int_{M_h} (\partial_j \bar{u} \partial_i u) D(\psi)^{ij}$$

with

$$D(\psi)^{ij} = \frac{2}{\sqrt{\det(h)}} \partial_l \left(\sqrt{\det(h)} h^{ij} h^{kl} \partial_k \psi \right) - 4h^{il} \partial_l (h^{kj} \partial_k \psi).$$

As Θ is real-valued, so is $(\phi \partial_t)^2 \Theta$, and hence

$$\begin{aligned} (\phi \partial_t)^2 \Theta = & - \int_{M_h} \left(\frac{1}{2} (\Delta_h^2 \psi) + \frac{1}{4} \nabla_h \psi \cdot \nabla_h \mathcal{R}_h \right) |u|^2 + \int_{M_h} (\Delta_h \psi) \nabla_h u \cdot \nabla_h u \\ & - \frac{1}{2} \operatorname{Re} \int_{M_h} (\partial_j \bar{u} \partial_i u) D(\psi)^{ij}. \end{aligned}$$

With some extra computations, we get

$$(\phi \partial_t)^2 \Theta = - \int_{M_h} \left(\frac{1}{2} (\Delta_h^2 \psi) + \frac{1}{4} \nabla_h \psi \cdot \nabla_h \mathcal{R}_h \right) |u|^2 + \frac{1}{2} \operatorname{Re} \int_{M_h} (\partial_j \bar{u} \partial_i u) D_1(\psi)^{ij}$$

with

$$D_1(\psi)^{ij} = 4h^{il} h^{kj} \partial_l \partial_k \psi + 2\partial_k \psi (-h^{kl} \partial_l h^{ij} + 2h^{il} \partial_l h^{kj}).$$

Thanks to the real part, we have a symmetry in i and j . Indeed,

$$\operatorname{Re}(\partial_j \bar{u} \partial_i u D_1(\psi)^{ij}) = \operatorname{Re}(\partial_i \bar{u} \partial_j u D_1(\psi)^{ij}).$$

Thus, we can replace $D_1(\psi)^{ij}$ by $\frac{1}{2}(D_1(\psi)^{ij} + D_1(\psi)^{ji})$, which yields

$$\frac{1}{2} \operatorname{Re} \int_{M_h} (\partial_j \bar{u} \partial_i u) D_1(\psi)^{ij} = 2 \int_{M_h} (\partial_j \bar{u} \partial_i u) D^2(\psi)^{ij}$$

with

$$D^2(\psi)^{ij} = h^{il} h^{kj} \partial_l \partial_k \psi + \frac{1}{2} \partial_k \psi (-h^{kl} \partial_l h^{ij} + h^{il} \partial_l h^{kj} + h^{jl} \partial_l h^{ki}).$$

We recognize the affine connection

$$\Lambda^{k,ij} = h^{il} h^{jm} \Lambda_{lm}^k = \frac{1}{2} (h^{kl} \partial_l h^{ij} - h^{il} \partial_l h^{kj} - h^{jl} \partial_l h^{ki}),$$

which yields

$$(4.8) \quad D^2(\psi)^{ij} = h^{il} h^{kj} \partial_l \partial_k \psi - \Lambda^{k,ij} \partial_k \psi.$$

Finally, we get the virial identity,

$$\begin{aligned} & - \int_{M_h} \left(\frac{1}{2} (\Delta_h^2 \psi) + \frac{1}{4} \nabla_h \psi \cdot \nabla_h \mathcal{R}_h \right) |u|^2 + 2 \int_{M_h} (\partial_j \bar{u} \partial_i u) D^2(\psi)^{ij} \\ & = -(\phi \partial_t) \operatorname{Re} \left(\int_{M_h} (\Delta_h \psi) \bar{u} \phi \partial_t u + 2 \int_{M_h} \nabla_h \psi \cdot \nabla_h \bar{u} \phi \partial_t u \right). \end{aligned}$$

□

5. THE ASYMPTOTICALLY FLAT CASE

The proof of Theorem (1.1) is fairly classical in this setting (see e.g. [4], [8]); nevertheless, before getting into details, let us give a brief sketch of it in order to make the various steps easier to be followed. The idea is to rely on virial identity (4.6), plug in it a proper choice of the multiplier (with a fixed $R > 0$) that we will define in the next subsection, integrate in time and carefully estimate all the terms. We will start from the Right Hand Side: making use of the modified Hardy inequality 2.1 will allow us to estimate from above with some energy-type terms at some fixed times 0 and T . Then, we will have to bound the Left Hand Side from below, which will be significantly more technical. Here we will make heavy use of our asymptotic-flatness (and smallness) assumptions to prove the estimate of the single terms and, roughly speaking, treat the non-flat ones as perturbations. To absorb them, it will be necessary to take the sup in $R > 0$: this will prevent us from exchanging the time and space norms in the Left Hand Side of (1.9). Eventually, we will take the sup in time and use conservation of energy.

5.1. Choice of the multiplier. We here present the multiplier function ψ that will be used in the main proof. We define the radial function $\psi(x)$ as

$$\psi_0(x) = \int_0^{|x|} \psi'_0(s) ds$$

where

$$\psi'_0(r) = \begin{cases} \frac{r}{3}, & r \leq 1 \\ \frac{1}{2} - \frac{1}{6r^2}, & r > 1 \end{cases}$$

(with a slight abuse we will use the same notation for $\psi(x)$ and $\psi(|x|)$). We then define the scaled function

$$\psi_R(r) := R\psi_0\left(\frac{r}{R}\right), \quad R > 0$$

for which we have

$$\psi'_R(r) = \begin{cases} \frac{r}{3R}, & r \leq R \\ \frac{1}{2} - \frac{R^2}{6r^2}, & r > R. \end{cases}$$

Moreover,

$$\psi''_R(r) = \begin{cases} \frac{1}{3R}, & r \leq R \\ \frac{R^2}{3r^3}, & r > R. \end{cases}$$

$$\psi'''_R(r) = -\frac{R^2}{r^4} \mathbf{1}_{|x| \geq R},$$

$$\psi_R^{iv} = 4\frac{R^2}{r^5} \mathbf{1}_{|x| \geq R} - \frac{1}{R^2} \delta_{|x|=R}.$$

Notice that, for every $|x| \geq 0$,

$$\psi_R''(|x|) \leq \frac{1}{2 \max\{R, |x|\}} \leq \frac{1}{2|x|}, \quad \psi_R'(|x|) \leq \frac{1}{2}.$$

In the following we shall simply denote with $\psi = \psi_R$ for a fixed R .

Remark 5.1. Our choice of the multiplier ψ here is classical, and already highly used in several papers to prove smoothing estimates for different dispersive equations, also in some perturbative settings. The function ψ is a mix of the so called *virial* and *Morawetz* multipliers, which are respectively given by $|x|$ and $|x|^2$; originally, the choice of such a function was dictated by the conditions of having a positive bi-laplacian and a negative Hessian, together with some good decay at infinity. Of course, in a fully variable coefficients setting, this properties are much more difficult to be fulfilled, and a smart choice of the multiplier, to the best of our knowledge, has never been attempted in this general case. Therefore our choice is motivated by perturbative arguments: the idea of the proof will be that the "leading" terms in the inequality will mainly recover the ones in the flat case, while the terms heavily involving the variable coefficients will be instead of perturbative nature.

5.2. Estimate of the right hand-side (RHS). We use the Dirac equation (1.7) to rewrite the right hand side of (4.6) as (notice that the mass term vanishes when taking the real part)

$$(5.1) \quad (\phi \partial_t) \operatorname{Re} \left(i \int_{M_h} (\Delta_h \psi) H u \bar{u} + 2 \int_{M_h} \nabla_h \psi \cdot \nabla_h \bar{u} H u \right).$$

First of all, by the application of Young's inequality we can write the estimate

$$(5.2) \quad \left| \int_{M_h} [(\Delta_h \psi) H u \bar{u} + 2 \nabla_h \psi \cdot \nabla_h \bar{u} H u] \right| \leq \frac{3}{2} \|H u\|_{L^2_{M_h}}^2 + \|\nabla_h \psi \cdot \nabla_h \bar{u}\|_{L^2_{M_h}}^2 + \frac{1}{2} \|\Delta_h \psi \bar{u}\|_{L^2_{M_h}}^2.$$

Recalling (2.2), (2.8) and (2.12) we then have

$$(5.3) \quad \begin{aligned} \|\Delta_h \psi \bar{u}\|_{L^2_{M_h}}^2 &\leq \frac{3N}{4} \left\| \frac{u}{|x|} \right\|_{L^2_{M_h}}^2 + \frac{C_\nabla}{4\nu^{3/2}} \left\| \frac{u}{\langle x \rangle^{1+\alpha}} \right\|_{L^2_{M_h}}^2 \\ &\leq C_H \left(\frac{3N}{4} + \frac{C_\nabla N^{3/2}}{4} \right) \|H u\|_{L^2_{M_h}}^2. \end{aligned}$$

Moreover, due to condition (2.10), we have

$$|\nabla_h \psi \cdot \nabla_h \bar{u}| \leq N/2 |\nabla u|,$$

so that applying estimate (2.16) with the choice $\varepsilon = 1 - \frac{4C(\sigma)}{\nu^4}$, we have

$$(5.4) \quad \|\nabla_h \psi \cdot \nabla_h \bar{u}\|_{L^2_{M_h}}^2 \leq \frac{1}{2} \frac{N}{1 - \frac{4C(\sigma)}{\nu^4}} \|H u\|_{L^2_{M_h}}^2$$

Therefore, multiplying (5.3) times ϕ^{-1} and integrating in time between 0 and T we obtain plugging (5.3) and (5.4) into (5.2)

$$(5.5) \quad \left| \int_0^T \phi^{-1}(t) (\phi(t) \partial_t) \int_{M_h} [(\Delta_h \psi) H u \bar{u} + 2 \nabla_h \psi \cdot \nabla_h \bar{u} H u] \right| \\ \lesssim_{\nu, N, \sigma} \|Hu(T)\|_{L^2_{M_h}}^2 + \|Hu(0)\|_{L^2_{M_h}}^2.$$

5.3. Estimate of the left hand-side (LHS). We now deal with the left hand side of identity (4.6). We estimate each term separately, and start with the one involving the gradient, namely

$$(5.6) \quad 2 \int_{M_h} (\partial_j \bar{u} \partial_i u) D^2(\psi)^{ij}.$$

Recalling (4.8), we treat separately terms involving derivatives on the coefficients from the others. Concerning $\Lambda^{k,ij}$ we have

$$|(\partial_j \bar{u} \partial_i u) \Lambda^{k,ij}| \leq 3 |h_{inv}| |h'_{inv}| |\nabla u|^2 |\psi'|$$

and thus, by our assumptions (2.10) and from the bound on ψ' ,

$$(5.7) \quad |(\partial_j \bar{u} \partial_i u) \Lambda^{k,ij}| \leq \frac{3}{2} N C_h \langle x \rangle^{-1-\sigma} |\nabla u|^2.$$

Turning to the other term, we use the fact that ψ is radial to rewrite it as follows

$$h^{il} h^{kj} \partial_l \partial_k \psi = h^{il} h^{kj} \hat{x}_l \hat{x}_k \left(\psi'' - \frac{\psi'}{|x|} \right) + h^{il} h^{jl} \frac{\psi'}{|x|}.$$

We restrict the quantity above first in the region $|x| \leq R$ where, notice, $\psi'' = \frac{\psi'}{|x|}$. Therefore,

$$(5.8) \quad \mathbf{1}_{|x| \leq R} (\partial_j \bar{u} \partial_i u) h^{il} h^{kj} \partial_l \partial_k \psi = \frac{1}{3R} \mathbf{1}_{|x| \leq R} h^{il} h^{jl} (\partial_j \bar{u} \partial_i u) \\ \geq \frac{\nu^2}{3R} \mathbf{1}_{|x| \leq R} |\nabla u|^2$$

where in the last inequality we have used (2.6). In the region $|x| > R$ we have instead

$$(5.9) \quad \mathbf{1}_{|x| > R} h^{il} h^{kj} \partial_l \partial_k \psi = \frac{1}{2|x|} [h^{il} h^{jl} - h^{il} h^{kj} \hat{x}_l \hat{x}_k] + \frac{R^2}{2|x|^3} \hat{x}_l \hat{x}_k h^{ik} h^{jl} - \frac{R^2}{6|x|^3} h^{il} h^{jl} \geq 0$$

in the sense of matrices (notice that $h^{il} h^{jl} - h^{il} h^{kj} \hat{x}_l \hat{x}_k \geq 0$ in the sense of matrices). We can therefore neglect this term.

We thus multiply (5.6) times ϕ^{-1} , and integrate in time between 0 and T . Exchanging the integrals and applying (5.7), (5.8) and (5.9) therefore gives (recall

(2.17))

$$(5.10) \quad 2 \int_{M_h} \phi^{-1} \int_0^T (\partial_j \bar{u} \partial_i u) D^2 \psi^{ij} \geq \frac{2\nu^2}{3R} \int_{M_h \cap B_R} \|\nabla u\|_{L_{\phi,T}^2}^2 - C_{D^2} \|\nabla u\|_{Y L_{\phi,T}^2}^2$$

with the constant

$$(5.11) \quad C_{D^2} = \frac{12\nu^{3/2} C_h}{\sqrt{N}\sigma}$$

Now we turn to the bi-laplacian term, that is

$$(5.12) \quad \frac{1}{2} \int_{M_h} (\Delta_h^2 \psi) |u|^2.$$

First of all observe that

$$\Delta_h(fg) = (\Delta_h f)g + 2\nabla_h f \cdot \nabla_h g + (\Delta_h g)f,$$

so that we can write, after some manipulations

$$\Delta_h^2 \psi = \Delta_h \Delta_h \psi = I + II + III + IV$$

with

$$I = \hat{h} \cdot \Delta_h \psi'' + (\bar{h} - \hat{h}) \Delta \left(\frac{\psi'}{|x|} \right),$$

$$II = A \hat{h} \cdot \psi'' + A(\bar{h} - \hat{h}) \cdot \frac{\psi'}{|x|},$$

$$III = 2\nabla_h \hat{h} \cdot \nabla_h \psi'' + 2\nabla_h (\bar{h} - \hat{h}) \cdot \nabla_h \frac{\psi'}{|x|},$$

$$IV = \Delta_h \left(\frac{1}{\sqrt{\det(h)}} \partial_j (\tilde{h}^{jk} \hat{x}_k \psi') \right).$$

We separate terms involving derivatives on the coefficients of h^{jk} (which will be of perturbative nature) from the others. After some long winded but not difficult computations (see [8] Section 4.4 for further details) one gets

$$\Delta_h^2 \psi = S(x) + R(x)$$

$$\begin{aligned} S(x) = & \hat{h}^2 \psi^{iv} + 2\hat{h}(\bar{h} - \hat{h}) \frac{\psi'''}{|x|} + \frac{(\bar{h} - \hat{h})(\bar{h} - 3\hat{h})}{|x|^2} \left(\psi'' - \frac{\psi'}{|x|} \right) + \\ & + \frac{2}{|x|^2} [h^{\ell m} h^{\ell m} - \bar{h} \hat{h} - 4(|h\hat{x}|^2 - \hat{h}^2)] \left(\psi'' - \frac{\psi'}{|x|} \right) + \\ & + \frac{4}{|x|} [|h\hat{x}|^2 - \hat{h}^2] \left(\psi''' - \frac{\psi''}{|x|} + \frac{\psi'}{|x|^2} \right) \end{aligned}$$

and

$$\begin{aligned}
\sqrt{\det(h)}R(x) = & \hat{h}\partial_m(\tilde{h}^{\ell m})\hat{x}_m\psi''' + (\bar{h} - \hat{h})\partial_k(\tilde{h}^{jk})\hat{x}_k\left(\frac{\psi''}{|x|} - \frac{\psi'}{|x|^2}\right) + \\
& + [\partial_j(\tilde{h}^{jk}\partial_k(h^{\ell m})\hat{x}_\ell\hat{x}_m) + \partial_j(\tilde{h}^{jk}h^{\ell m})\partial_k(\hat{x}_\ell\hat{x}_m)]\left(\psi'' - \frac{\psi'}{|x|}\right) \\
& + \sqrt{\det(h)}(\Delta_h\bar{h})\frac{\psi'}{|x|} + 2\sqrt{\det(h)}h^{jk}\partial_k h^{\ell m}\hat{x}_\ell\hat{x}_m\hat{x}_j\left(\psi''' - \frac{\psi''}{|x|}\right) + \\
& + 2\sqrt{\det(h)}h(\nabla\bar{h}, \nabla\frac{\psi'}{|x|}) + \sqrt{\det(h)}\Delta_h\left(\frac{1}{\sqrt{\det(h)}}\partial_j(\tilde{h}_{jk}\hat{x}_k\psi')\right).
\end{aligned}$$

In our assumptions on the metric h and noticing that by the definition of ψ we have

$$|\psi'| \leq \frac{|x|}{2(R\vee|x|)}, \quad |\psi''| \leq \frac{n-1}{2n(R\vee|x|)}, \quad |\psi'''| \leq \frac{n-1}{2(R\vee|x|)|x|},$$

the remainder term $R(x)$ can be estimated as

$$(5.13) \quad |R(x)| \leq \frac{36C_h(N + C_h)}{|x|\langle x \rangle^{1+\sigma} \max\{R, |x|\}}.$$

For $S(x)$ we have the equality :

$$\begin{aligned}
S(x) = & \hat{h}^2\psi^{iv} + \left(2\bar{h}\hat{h} - 6\hat{h}^2 + 4|h\hat{x}|^2\right)\frac{\psi'''}{|x|} \\
& + \left(2h^{\ell m}h^{\ell m} + \bar{h}^2 - 6\bar{h}\hat{h} + 15\hat{h}^2 - 12|h\hat{x}|^2\right)\left(\frac{\psi''}{|x|^2} - \frac{\psi'}{|x|^3}\right).
\end{aligned}$$

As we can write $h(x) = I + \varepsilon(x)$ (meaning $\varepsilon_{jk} = h^{jk} - \delta_{jk}$), we have

$$h^{\ell m}h^{\ell m} = \delta_{\ell m}\varepsilon_{\ell m} + 2\delta_{\ell m}\varepsilon_{\ell m} + \varepsilon_{\ell m}\varepsilon_{\ell m} = 3 + 2\bar{\varepsilon} + \varepsilon_{\ell m}\varepsilon_{\ell m}$$

as well as

$$\hat{h} = 1 + \hat{\varepsilon}, \quad \bar{a} + \bar{h}, \quad |h\hat{x}|^2 = 1 + 2\hat{\varepsilon} + |\varepsilon\hat{x}|^2.$$

Notice also that by assumption $|\varepsilon(x)| = |h(x) - I| \leq C\langle x \rangle^{-\sigma} < 1$ and therefore

$$|\bar{\varepsilon}| \leq 3C_I\langle x \rangle^{-\sigma}, \quad |\hat{\varepsilon}| \leq C_I\langle x \rangle^{-\sigma}, \quad |\varepsilon\hat{x}| \leq C_I\langle x \rangle^{-\sigma}$$

so that

$$\begin{aligned}
2h^{\ell m}h^{\ell m} + \bar{h}^2 - 6\bar{h}\hat{h} + 15\hat{h}^2 - 12|h\hat{x}|^2 &= 4\bar{\varepsilon} - 12\hat{\varepsilon} + 2\varepsilon_{\ell m}\varepsilon_{\ell m} + \bar{\varepsilon}^2 \\
&\quad - 6\bar{\varepsilon}\hat{\varepsilon} + 15\hat{\varepsilon}^2 - 12|\varepsilon\hat{x}|^2 \\
&\geq 4\bar{\varepsilon} - 12\hat{\varepsilon} - 6\bar{\varepsilon}\hat{\varepsilon} - 12|\varepsilon\hat{x}|^2 \\
&\geq -46C_I\langle x \rangle^{-\sigma}.
\end{aligned}$$

Also, as $1 - C_I \leq \hat{h} \leq 1 + C_I$,

$$-\hat{h}^2 \leq -(1 - C_I)^2, \quad (3\hat{h} - \bar{h})\hat{h} \leq 6C_I(1 + C_I) \leq 12C_I.$$

Therefore, under our assumptions and with our choice of the multiplier ψ , we obtain the estimates

$$S(x) \leq -(1 - C_I)^2 \frac{1}{R^2} \delta_{|x|=R} \quad \text{for } |x| \leq R,$$

and

$$S(x) \leq 24C_I \left[\frac{R^5}{|x|^5} + \frac{1}{|x|^3 \langle x \rangle^\sigma} \right] \quad \text{for } |x| > R.$$

We now multiply times ϕ^{-1} and integrate in time (5.12) from 0 to T : this gives

$$-\int_0^T \phi^{-1} \int_{M_h} \Delta_h^2 \psi |u|^2 = -\int_{M_h} \Delta_h^2 \psi \|u\|_{L_{\phi,T}^2}^2 = I + II$$

with

$$I = -\int_{M_h} S(x) \|u\|_{L_{\phi,T}^2}^2, \quad II = -\int_{M_h} R(x) \|u\|_{L_{\phi,T}^2}^2$$

and estimate the two terms separately. For the $S(x)$ term we get, thanks to (2.18) and (2.19),

$$I \geq (1 - C_I)^2 \frac{1}{R^2} \int_{M_h \cap S_R} \|u\|_{L_{\phi,T}^2}^2 - \frac{72C_I}{\sigma} \left(\frac{\nu}{N} \right)^{3/2} \|u\|_{XL_{\phi,T}^2}^2.$$

The $R(x)$ term can be instead estimated with

$$II \geq -36C_h(N + C_h) \int_0^T \left[\int_{M_h \cap B_R} + \int_{M_h \cap B_R^c} \right] \frac{|u|^2}{\phi(t)|x|^2 \langle x \rangle^{1+\sigma}}$$

where B_R^c is the complementary set of B_R , that is the region where $r > R$. Thanks to (2.19) we have

$$(5.14) \quad \int_0^T \int_{M_h \cap B_R} \frac{|u|^2}{\phi(t)|x|^2 \langle x \rangle^{1+\sigma}} \leq \int_{M_h \cap B_R} \frac{\|u\|_{L_{\phi,T}^2}^2}{|x|^2 \langle x \rangle^{1+\sigma}} \leq \frac{2}{\sigma} \left(\frac{\nu}{N} \right)^{3/2} \|u\|_{XL_{\phi,T}^2}^2$$

and thanks to (2.5) and (2.22)

$$(5.15) \quad \int_0^T \int_{M_h \cap B_1} \frac{|u|^2}{\phi(t)|x|^2 \langle x \rangle^{1+\sigma}} \leq \int_0^T \int_{M_h \cap B_1} \frac{|u|^2}{\phi(t)|x|^2} \leq 4 \|\nabla u\|_{L^2(M_h \cap B_1)L_{\phi,T}^2}^2.$$

From (5.14) and (5.15) we thus obtain

$$II \geq -\frac{324C_h(N + C_h)}{\sigma} \left(\frac{\nu}{N} \right)^{3/2} \left(\|u\|_{XL_{\phi,T}^2}^2 + \|\nabla u\|_{L^2(M_h \cap B_1)L_{\phi,T}^2}^2 \right).$$

Putting all together gives

$$\begin{aligned} & -\frac{1}{2} \int_0^T \phi^{-1} \int_{M_h} \Delta_h^2 \psi |u|^2 \geq \frac{(1 - C_I)^2}{2} \frac{1}{R^2} \int_{M_h \cap S_R} \|u\|_{L_{\phi,T}^2}^2 \\ & -\frac{1}{\sigma} \left(\frac{\nu}{N} \right)^{3/2} \left[36C_I \|u\|_{XL_{\phi,T}^2}^2 + 162C_h(N + C_h) \left(\|u\|_{XL_{\phi,T}^2}^2 + \|\nabla u\|_{L^2(M_h \cap B_1)L_{\phi,T}^2}^2 \right) \right] \end{aligned}$$

Recalling (2.5) eventually gives

$$(5.16) \quad -\frac{1}{2} \int_0^T \phi(t)^{-1} \int_{M_h} \Delta_h^2 \psi |u|^2 \geq \frac{(1-C_I)^2}{2} \frac{1}{R^2} \int_{M_h \cap S_R} \|u\|_{L_{\phi,T}^2}^2 \\ - C_{\Delta^2}^I \|u\|_{XL_{\phi,T}^2}^2 - C_{\Delta^2}^{II} \|\nabla u\|_{YL_{\phi,T}^2}^2.$$

where the constants are explicitly given by

$$(5.17) \quad C_{\Delta^2}^I = \left(\frac{\nu}{N}\right)^{3/2} \frac{36C_I + 162C_h(N + C_h)}{\sigma}, \quad C_{\Delta^2}^{II} = \left(\frac{\nu}{N}\right)^{3/2} \frac{162C_h(N + C_h)}{\sigma}$$

We now turn to the last term of (4.6) that is

$$(5.18) \quad \int_{M_h} \nabla_h \psi \cdot \nabla_h \mathcal{R}_h |u|^2.$$

Notice that it involves only terms with derivatives on h (and indeed vanishes in the flat case). Therefore, using repeatedly assumptions (2.10), it is not difficult to show that

$$(5.19) \quad |\nabla \mathcal{R}_h(x)| \leq \frac{3C_h(1 + 18N^2)}{\langle x \rangle^{3+3\sigma}} + \frac{3NC_h^2 + 9NC_h^2 + 9N^3C_h^2}{\langle x \rangle^{2+2\sigma}|x|} + \frac{3N^2C_h^3}{\langle x \rangle^{1+\sigma}|x|^2} \\ \leq \frac{C_{\mathcal{R}}}{\langle x \rangle^{1+\sigma}|x|^2}.$$

where the constant $C_{\mathcal{R}}$ is the sum of the three numerators above, that is

$$C_{\mathcal{R}} = 3C_h(1 + 18N^2) + 3NC_h^2(4 + 3N^2) + 3C_h^3N^2.$$

We now multiply as usual (5.18) times ϕ^{-1} and integrate in time between 0 and T : following calculations and relying on (2.19)-(2.22) yield the estimate

$$(5.20) \quad \int_0^T \phi(t)^{-1} \int_{M_h} \nabla_h \psi \cdot \nabla_h \mathcal{R}_h |u|^2 \geq -C_{\mathcal{R}} \int_0^T \left[\int_{M_h \cap B_R} + \int_{M_h \cap B_R^c} \right] \frac{|u|^2}{\phi(t)|x|^2 \langle x \rangle^{1+\sigma}} \\ \geq -C_{\mathcal{R}} \left[\frac{2}{\sigma} \left(\frac{\nu}{N}\right)^{3/2} \|u\|_{XL_{\phi,T}^2}^2 + 2 \left(\frac{\nu}{N}\right)^{3/2} \|\nabla u\|_{YL_{\phi,T}^2}^2 \right] \\ = -4C_{\mathcal{R}}^I \|u\|_{XL_{\phi,T}^2}^2 - C_{\mathcal{R}}^{II} \|\nabla u\|_{YL_{\phi,T}^2}^2$$

with

$$(5.21) \quad C_{\mathcal{R}}^I = \frac{C_{\mathcal{R}}}{2\sigma} \left(\frac{\nu}{N}\right)^{3/2}, \quad C_{\mathcal{R}}^{II} = \frac{C_{\mathcal{R}}}{2} \left(\frac{\nu}{N}\right)^{3/2}.$$

5.4. Conclusion of the proof. We multiply times ϕ^{-1} and integrate in time identity (4.6) from 0 to T , exchange integrals and use (5.10), (5.16), (5.20) for the left hand side and (5.5) for the right hand side to obtain

$$(5.22) \quad \frac{(1-C_I)^2}{2} \frac{1}{R^2} \int_{M_h \cap S_R} \|u\|_{L_{\phi,T}^2}^2 + \frac{2\nu^2}{3R} \int_{M_h \cap B_R} \|\nabla v\|_{L_{\phi,T}^2}^2$$

$$\begin{aligned}
& -(C_{\Delta^2}^I + C_{\mathcal{R}}^I) \|u\|_{XL_{\phi,T}^2}^2 - (C_{D^2} + C_{\Delta^2}^{II} + C_{\mathcal{R}}^{II}) \|\nabla u\|_{YL_{\phi,T}^2} \\
& \leq C_{\nu,N,\sigma} \|Hu(T)\|_{L_{M_h}^2}^2 + \|Hu(0)\|_{L_{M_h}^2}^2
\end{aligned}$$

where the constants are explicit and given by (5.11), (5.17) and (5.21). We also stress that the constant $C = C_{\nu,N,\sigma}$ does not depend on R . We now take the sup over $R > 1$ on the left hand side of (5.22) (notice that only the first two terms of inequality above depend on R). We use (2.20) to estimate, for $0 < \theta < 1$,

$$\begin{aligned}
(5.23) \quad & \frac{(1 - C_I)^2}{2} \sup_{R>1} \frac{1}{R^2} \int_{M_h \cap S_R} \|u\|_{L_{\phi,T}^2}^2 \geq (1 - \theta) \frac{(1 - C_I)^2}{2} \sup_{R>1} \frac{1}{R^2} \int_{M_h \cap S_R} \|u\|_{L_{\phi,T}^2}^2 \\
& + \theta (1 - C_I)^2 \left[\frac{1}{4} \left(\frac{\nu}{N} \right)^{3/2} \|u\|_{XL_{\phi,T}^2}^2 - \frac{13}{4} \|\nabla u\|_{YL_{\phi,T}^2}^2 \right].
\end{aligned}$$

Thanks to our assumption (2.9), we can take $\nu = 1 - C_I$, such that

$$\sup_{R>1} \frac{2\nu^2}{3R} \int_{M_h \cap B_R} \|\nabla u\|_{L_{\phi,T}^2}^2 \geq \frac{2}{3} (1 - C_I)^2 \|\nabla u\|_{YL_{\phi,T}^2}^2.$$

Choosing θ in (5.23) such that $\frac{13\theta}{4} \leq \frac{2}{3}$ (e.g. $\theta = 1/5$) and using the simple property

$$\sup_R (F_1(R) + F_2(R)) \geq \frac{1}{2} \left(\sup_R F_1(R) + \sup_R F_2(R) \right)$$

therefore yields

$$\begin{aligned}
& \frac{(1 - C_I)^2}{2} \sup_{R>1} \frac{1}{R^2} \int_{M_h \cap S_R} \|u\|_{L_{\phi,T}^2}^2 + \sup_{R>1} \frac{2\nu^2}{3R} \int_{M_h \cap B_R} \|\nabla u\|_{L_{\phi,T}^2}^2 \\
& \geq (1 - C_I)^2 \left(\frac{1}{40} \left(\frac{\nu}{N} \right)^{3/2} \|u\|_{XL_{\phi,T}^2}^2 + \frac{1}{120} \|\nabla u\|_{YL_{\phi,T}^2}^2 \right)
\end{aligned}$$

which plugged into (5.22) finally gives

$$M_1 \|u\|_{XL_{\phi,T}^2}^2 + M_2 \|\nabla u\|_{YL_{\phi,T}^2}^2 \leq C_{\nu,N,\sigma} \|Hu(T)\|_{L_{M_h}^2}^2 + \|Hu(0)\|_{L_{M_h}^2}^2$$

with

$$M_1 = \frac{(1 - C_I)^2}{40} - C_{\Delta^2}^I - C_{\mathcal{R}}^I$$

and

$$M_2 = \frac{(1 - C_I)^2}{120} - C_{D^2} - C_{\Delta^2}^{II} - C_{\mathcal{R}}^{II}.$$

The proof is concluded provided the constants M_1 and M_2 are positive, i.e. if the constants C_I and C_h are small enough, by letting T to infinity and using the conservation of the L^2 -norm of Hu , which is standard.

6. WRAPPED PRODUCTS

We dedicate this section to prove Theorem 1.2. First of all, we notice that if h is in the form (1.11) the following result holds.

Proposition 6.1. *Let h be a wrapped product. We have*

$$(6.1) \quad \mathcal{R}_h = -2\frac{d''}{d} + \frac{1}{2}\left(\frac{d'}{d}\right)^2 + \frac{1}{d}\mathcal{R}_\beta$$

and

$$(6.2) \quad \Lambda^{1,ij} = 0 \text{ if } i = 1 \text{ or } j = 1 \text{ and } \Lambda^{1,ij} = -\frac{1}{2}\frac{d'}{d^2}\beta^{ij} \text{ otherwise.}$$

Proof. The proof is straightforward computation. □

The strategy to prove Theorem 1.2 is the same we have seen in details in the previous section to deal with the asymptotically flat case, and thus consists in applying the virial identity (4.6) to an appropriate function φ , and then estimate the single terms. We deal with three different cases separately.

6.1. Hyperbolic-type metrics. We start with the choice $d(r) = e^{r/2}$ that, as one may re-scale, includes some hyperbolic manifolds. In this case we have

$$\mathcal{R}_h = -\frac{3}{8} + e^{-r/2}\mathcal{R}_\beta, \quad \partial_1\mathcal{R}_h = -\frac{1}{2}e^{-r/2}\mathcal{R}_\beta, \quad \Lambda^{1,ij} = -\frac{1}{4}h^{ij}.$$

We recall that under the hypothesis of Theorem 1.2 for the hyperbolic type metrics, the curvature of β is non-negative, that is $\mathcal{R}_\beta \geq 0$.

We make the following choice for ψ_R , it is a function depending only on $x^1 = r$ and, with the notation $\psi'_R = \partial_1\psi_R$, we take

$$(6.3) \quad \psi'_R(r) = \begin{cases} 1 + \frac{r}{\langle R \rangle} & \text{if } r \leq R \\ 1 + \frac{2+R}{\langle R \rangle} - \frac{2e^{(R-r)/2}}{\langle R \rangle} & \text{if } r > R. \end{cases}$$

With this choice, we have that ψ is \mathcal{C}^2 and the following identities hold

$$\begin{aligned} \psi''_R &= \frac{1}{\langle R \rangle}1_{r \leq R} + \frac{2}{\langle R \rangle}e^{R-r}1_{r > R} \\ \Delta\psi_R &= \frac{\langle R \rangle + 2 + r}{2\langle R \rangle}1_{r \leq R} + 1_{r > R} \frac{\langle R \rangle + 2 + R}{2\langle R \rangle} \\ \Delta^2\psi_R &= -\frac{1}{2\langle R \rangle}\delta(r - R) + 1_{r \leq R} \frac{1}{4\langle R \rangle} \end{aligned}$$

where δ is the Dirac delta. We deduce from that $-\nabla_h \mathcal{R}_h \cdot \nabla_h \psi_R \geq 0$ and

$$-\int_{M_h} \Delta_h^2 \psi_R |u|^2 \geq \frac{1}{2\langle R \rangle} \int_{S_R} e^{R/2} |u|^2 d\beta - \frac{1}{4} \sup_{r \in \mathbb{R}_+} e^{r/2} \langle r \rangle^{-1} \int_{S_r} |u|^2 d\beta.$$

By straightforward computations again, we get that

$$\int D^2(\psi_R)^{ij} \partial_i \bar{u} \partial_j u \geq \frac{1}{4\langle R \rangle} \int_{B_R} |\nabla_h u|^2 dh.$$

By taking the supremum over R we get the following lemma.

Lemma 6.2. *Let $\eta_1, \eta_2 > 0$, there exists $C_{\eta_1, \eta_2} > 0$ such that for all u , we have*

$$\begin{aligned} & \|\langle r \rangle^{-(1+\eta_1)} u\|_{L^2(M_h)}^2 + \|\langle r \rangle^{-(1/2+\eta_2)} \nabla_h u\|_{L^2}^2 \leq \\ & C_{\eta_1, \eta_2} \sup_R \left(- \int_{M_h} (\Delta_h^2 \psi_R + \frac{1}{2} \nabla_h \mathcal{R}_h \cdot \nabla_h \psi_R) |u|^2 + \int_{M_h} D^2(\psi)^{ij} \partial_i \bar{u} \partial_j u \right). \end{aligned}$$

Proof. The proof is essentially based on the fact that

$$\begin{aligned} & \|\langle r \rangle^{-(1+\eta_1)} u\|_{L^2(M_h)}^2 + \|\langle r \rangle^{-(1/2+\eta_2)} \nabla_h u\|_{L^2}^2 \leq \\ & C_{\eta_1, \eta_2} \sup_R \left(\frac{1}{4\langle R \rangle} e^{R/2} \int_{S_R} |u|^2 d\beta + \frac{1}{8\langle R \rangle} \int_{B_R} |u|^2 dh \right). \end{aligned}$$

□

Lemma 6.3. *Under the hypothesis of Theorem 1.2, there exists C such that for all u solution of the linear Dirac equation, we have the following estimate*

$$\left| \int_{M_h} \Delta \psi_R \bar{u} \phi \partial_t u + \int_{M_h} \nabla_h \psi_R \cdot \nabla_h \bar{u} \phi \partial_t u \right| \leq C \|Hu(t)\|_{L^2(M_h)}.$$

Proof. We use that $\phi \partial_t u = Hu$, that the Laplacian of ψ_R and its gradient are uniformly bounded in R and that the L^2 norm of the gradient is bounded by the L^2 norm of H as it has been explained above. Indeed, we use (2.13) with the additional information that under the hypothesis of Theorem 1.2, $\mathcal{R}_h + 4m^2$ is more than a non-negative constant, which explains the hypothesis $m^2 > \frac{3}{32}$. □

Combining Lemma 6.2 where we replace $|u|^2$ by $\int \phi^{-1} |u|^2 dt$, and Lemma 6.3 and the conservation of $\|Hu\|_{L^2}$, we get local smoothing, that is estimate (1.12) in Theorem 1.2.

6.2. Flat-type metrics. We now take $d(r) = r^2$ which includes the flat case. With this choice we have

$$\mathcal{R}_h = -2r^{-2} + r^{-2} \mathcal{R}_\beta, \partial_r \mathcal{R}_h = -2(\mathcal{R}_\beta - 2) \frac{1}{r^3}.$$

If $\mathcal{R}_\beta \geq 2$, then the computations are exactly the same as in the flat case, and thus lead to estimate (1.13) in Theorem 1.2.

6.3. Sub-flat type metrics. We consider another specific case, which is $d(r) = r^n$ with $2 - \sqrt{2} < n \leq \frac{4}{3}$. With this choice we have

$$(6.4) \quad \mathcal{R}_h = \frac{4n - 3n^2}{2r^2} + \frac{1}{r^n} \mathcal{R}_\beta \text{ and } \partial_r \mathcal{R}_h = -\frac{4n - 3n^2}{r^3} - \frac{n}{r^{n+1}} \mathcal{R}_\beta.$$

As $\partial_r \mathcal{R}_h$ is negative (as long as \mathcal{R}_β is non negative), we may have that it can compensate losses due to the bi-laplacian.

Let us take

$$\psi'_R = \begin{cases} \frac{r}{\langle R \rangle} & \text{if } r \leq R \\ \left(\frac{n+1}{n} - \frac{1}{n} \left(\frac{R}{r} \right)^n \right) \frac{R}{\langle R \rangle} & \text{if } r > R \end{cases}$$

Notice that with this choice $\psi_R \in \mathcal{C}^2$. Moreover, we note the following properties :

- for $r \leq R$, we have $\psi''_R = \frac{1}{\langle R \rangle}$, and $\Delta_h \psi_R = \frac{n+1}{\langle R \rangle}$,
- For $r > R$, we have $\psi''_R > 0$, hence $\frac{R}{\langle R \rangle} \leq \psi'_R \leq \frac{(n+1)R}{\langle R \rangle}$.

Besides,

$$\Delta_h \psi_R = (n+1) \frac{R}{r \langle R \rangle}.$$

From these relations we can deduce, after some computations,

$$\Delta_h^2 \psi_R = -\frac{(n+1)}{R \langle R \rangle} \delta(r - R) - \mathbf{1}_{r > R} R(n+1)(n-2)r^{-3} \langle R \rangle^{-1}.$$

Proof of estimate (1.14). We recall that we take \mathcal{R}_β minored by a certain constant depending on n . Under this assumption, we have that \mathcal{R}_h is non negative which ensures that $\|Hu\|_{L^2(M_h)}$ controls

$$\|u\|_{L^2(M_h)} + \|\nabla_h u\|_{L^2(M_h)}.$$

First of all, we consider the terms in $|u|^2$ in the virial identity (4.6) inside the ball of radius R . Since $\partial_1 \mathcal{R}_h \leq 0$ and $\partial_1 \psi_R \geq 0$, we have

$$(6.5) \quad -\int_{B_R} (\Delta_h^2 + \frac{1}{2} \partial_1 \mathcal{R}_h \psi'_R) |u|^2 \geq \frac{(n+1)}{\langle R \rangle} R^{n-1} \int_{S_R} |u|^2 d\beta.$$

We now turn to the terms in $|u|^2$ in the virial identity outside B_R . Since $\partial_1 \mathcal{R}_h \leq 0$, and $\psi'_R \geq 0$, we have

$$-\partial_1 \mathcal{R}_h \psi'_R \geq -\partial_1 \mathcal{R}_h \inf \psi'_R \geq -\partial_1 \mathcal{R}_h \frac{R}{\langle R \rangle}.$$

This yields, by replacing $\partial_1 \mathcal{R}_h$ by its value (6.4),

$$-\Delta_h^2 \psi_R - \frac{1}{2} \partial_1 \mathcal{R}_h \psi'_R \geq \frac{R}{\langle R \rangle r^3} \left(-\frac{n^2}{2} - 2 + n + nr^{2-n} \frac{\mathcal{R}_\beta}{2} \right).$$

Let

$$R_0 = \left(\left(\frac{n^2}{2} + 2 - n \right) \frac{2}{n \min \mathcal{R}_\beta} \right)^{1/(2-n)}.$$

Note that R_0 decreases when \mathcal{R}_β increases as $n \leq \frac{4}{3} < 2$.

For $r \geq R_0$ we have $-\Delta_h^2 \psi_R - \frac{1}{2} \partial_1 \mathcal{R}_h \psi'_R \geq 0$, hence

$$-\Delta_h^2 \psi_R - \frac{1}{2} \partial_1 \mathcal{R}_h \psi'_R \geq -\mathbf{1}_{r \leq R_0} \frac{R}{\langle R \rangle r^3} \left(\frac{n^2}{2} + 2 - n \right).$$

We multiply by $|u|^2$ and integrate to get

$$-\int_{B_R^c} (\Delta_h^2 \psi_R + \frac{1}{2} \partial_1 \mathcal{R}_h \psi'_R) |u|^2 \geq -\frac{R}{\langle R \rangle} \left(\frac{n^2}{2} + 2 - n \right) \int_R^{R_0} r^{n-3} dr \int_{S_r} |u|^2 d\beta$$

If $R \geq R_0$ the right hand side is 0 and otherwise, we have

$$\int_R^{R_0} r^{n-3} dr \int_{S_r} |u|^2 d\beta \leq \int_R^{R_0} r^{-2} \langle r \rangle dr \sup_y y^{n-1} \langle y \rangle^{-1} \int_{S_y} |u|^2 d\beta.$$

We have

$$\int_R^{R_0} r^{-2} \langle r \rangle \leq \langle R_0 \rangle \left(\frac{1}{R} - \frac{1}{R_0} \right).$$

Hence

$$\frac{R}{\langle R \rangle} \int_R^{R_0} \langle r \rangle r^{-2} dr \leq \langle R_0 \rangle.$$

Summing with (6.5), we get

$$(6.6) \quad -\int (\Delta_h^2 \psi_R + \frac{1}{2} \partial_1 \mathcal{R}_h \psi'_R) |u|^2 \geq \frac{(n+1)}{\langle R \rangle} R^{n-1} \int_{S_R} |u|^2 d\beta \\ - \langle R_0 \rangle \left(\frac{n^2}{2} + 2 - n \right) \sup_y y^{n-1} \langle y \rangle^{-1} \int_{S_y} |u|^2 d\beta.$$

We deal with the terms in $\nabla_h u$ in the left hand side of the virial identity.

We have

$$D^2(\psi_R)^{11} = \psi_R'' \text{ and } D^2(\psi_R)^{ij} = \frac{1}{2} \frac{n}{r} h^{ij}(\psi_R'),$$

from which we deduce that above R , we have

$$D^2(\psi_R)^{ij} \partial_i \bar{u} \partial_j u \geq 0$$

and under R ,

$$D^2(\psi_R)^{ij} \partial_i \bar{u} \partial_j u \geq \frac{n}{2\langle R \rangle} |\nabla_h u|^2.$$

Therefore,

$$(6.7) \quad 2 \int_{M_h} D^2(\psi_R) \partial_i \bar{u} \partial_j u \geq \frac{n}{\langle R \rangle} \int_{B_R} |\nabla_h u|^2.$$

Combining (6.6) and (6.7), we get

$$(6.8) \quad \sup_R \left(\int_{M_h} \left(\frac{1}{2} \Delta_h^2 \psi_R + \frac{1}{4} \nabla_h \psi_R \cdot \nabla_h \mathcal{R}_h \right) |u|^2 + 2 \int_{M_h} D^2(\psi_R) \partial_i \bar{u} \partial_j u \right) \geq \\ \sup_y \left(\frac{1}{2} (n+1) y^{n-1} \langle y \rangle^{-1} \int_{S_y} |u|^2 d\beta + \frac{n}{\langle y \rangle} \int_{B_y} |\nabla_h u|^2 \right) \\ - \frac{1}{2} \langle R_0 \rangle \left(\frac{n^2}{2} + 2 - n \right) \sup_y y^{n-1} \langle y \rangle^{-1} \int_{S_y} |u|^2 d\beta.$$

There exists $C_n > 0$ such that if $\min \mathcal{R}_\beta \geq C_n$ then $\langle R_0 \rangle$ is close enough to 1 to get $(n+1) - \langle R_0 \rangle (\frac{n^2}{2} + 2 - n) > 0$. This is where we use the hypothesis $n > 2 - \sqrt{2}$. We take \mathcal{R}_β within this range to obtain that for all $\eta_1, \eta_2 > 0$, there exists C_{n,η_1,η_2} such that,

$$\| \langle r \rangle^{-(3/2+\eta_1)} u \|_{L^2(M_g)}^2 + \| \langle r \rangle^{-(1/2+\eta_2)} \nabla_h u \|_{L^2(M_g)}^2 \leq \\ C_{n,\eta_1,\eta_2} \sup_R \left(\int_{M_h} \left(\frac{1}{2} \Delta_h^2 \psi_R + \frac{1}{4} \nabla_h \psi_R \cdot \nabla_h \mathcal{R}_h \right) |u|^2 + 2 \int_{M_h} D^2(\psi_R) \partial_i \bar{u} \partial_j u \right).$$

Since $\Delta_h \psi_R$ and ψ'_R are bounded, combining this estimate with the virial identity gives the result. \square

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